The Emergence of Chaos on Non-Uniformly Timed Systems

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ABSTRACT

This paper is motivated by the problems posed in control design when actuators, sensors, and/or computational nodes connect via unreliable or unpredictable communications channels. In these cases, non-uniformities are introduced into the underlying time domain of the system. Our central question is whether chaos can emerge in a system as a result of changing *only* the time domain. We answer the question in the positive by producing an example. Additionally, we find fractal structure in the emergence of chaos in this example.

KEY WORDS

Chaos, Hybrid Systems, Dynamic Equations on Time Scales, Network Control

I. Background

We use the theory of dynamic equations on time scales (DETS), a field of mathematics that has proven useful to describe, model and analyze systems that evolve on time domains other than continuous time or uniform discrete time.

The theory of time scales springs from the seminal paper of Stefan Hilger [1] in 1990. This work aimed to unify various overarching concepts from the (sometimes disparate) theories of discrete and continuous dynamical systems [2], and also to extend these theories to more general classes of dynamical systems. From there, time scales theory advanced fairly quickly, culminating in the introductory text [3] and the more advanced monograph [4].

A *time scale* $\mathbb{T}$ is any nonempty, (topologically) closed subset of the real numbers $\mathbb{R}$. Thus time scales can be (but are not limited to) any of the usual integer subsets (e.g. $\mathbb{Z}$ or $\mathbb{N}$), the entire real line $\mathbb{R}$, or any combination of discrete points unioned with closed intervals. For example, if $q > 1$ is fixed, the *quantum time scale* $\overline{q^{\mathbb{Z}}}$ is defined as $\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$. The quantum time scale appears throughout the mathematical physics literature, where the dynamical systems of interest are the q-difference equations [5], [6], [7]. Another interesting example is the *pulse time scale* $\mathbb{P}_{a,b}$ formed by a union of closed intervals each of length a and gap b: $\mathbb{P}_{a,b} := \bigcup_k [k(a+b), k(a+b)+a]$. This time scale is used to study duty cycles of various waveforms. A generalization of the pulse time scale is the *alternating pulse time scale* $\mathbb{P}_{a,b_1,b_2;n}$ formed by a union of closed intervals each of length a with n gaps of size b_1 followed by n gaps of size b_2 in between the closed intervals. We focus on the alternating pulse time scale for the example in this paper.

To date, the bulk of engineering systems theory rests on two time scales, $\mathbb{R}$ and $\mathbb{Z}$ (or more generally $h\mathbb{Z}$, meaning discrete points separated by distance h). However, there are occasions when necessity or convenience dictates the use of an alternate time scale. The question of how to approach the study of dynamical systems on time scales then becomes relevant, and in fact the majority of research on time scales so far has focused on expanding and generalizing the vast suite of tools available to the differential and difference equation theorist. We now briefly outline the portions of the time scales theory that are needed for this paper to be as self-contained as is practically possible.

The *forward jump operator* is given by $\sigma(t) := \inf_{s\in\mathbb{T}}\{s > t\}$ and the *graininess function* $\mu(t)$ by $\mu(t) := \sigma(t) - t$. The time scale derivative, $x^{\Delta}(t)$, is defined as

$$x^{\Delta}(t) := \lim_{\mu^*(t)\searrow\mu(t)} \frac{x(\sigma(t)) - x(t)}{\mu^*(t)}.$$

A benefit of this general approach is that the realms of differential equations and difference equations can now be viewed as special cases of more general *dynamic equations on time scales*, i.e. equations involving the delta derivative(s) of some unknown function. The upshot here is that the concepts in Table I apply just as readily to *any* closed subset of the real line as they do on $\mathbb{R}$ or $\mathbb{Z}$. Our goal is to leverage this general framework against wide classes of dynamical and control systems. Progress in this direction has been made in control [8], [9], [10], [11], [12], [13], [14], [15], and dynamic programming [16].

We close this section with the requisite tools from dynamical systems and chaos theory neccesary for this paper.

Definition 1. *The Lyapunov exponent, λ, of a sequence of points $\{x_n\} \subset \mathbb{R}$ is given by*

$$\lambda = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left|\frac{dx_{n+1}}{dx_n}\right|.$$

Definition 2. *A sequence of points $\{x_n\} \subset \mathbb{R}$ is a chaotic sequence if the sequence is bounded and has a positive Lyapunov exponent.*

TABLE I. CANONICAL TIME SCALES COMPARED TO THE GENERAL CASE.

	continuous	(uniform) discrete	time scale
domain	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{T}$
forward jump	$\sigma(t) \equiv t$	$\sigma(t) \equiv t+1$	$\sigma(t)$ varies
step size	$\mu(t) \equiv 0$	$\mu(t) \equiv 1$	$\mu(t)$ varies
differential operator	$\dot{x}(t) := \lim\limits_{h \to 0} \dfrac{x(t+h)-x(t)}{h}$	$\Delta x(t) := x(t+1) - x(t)$	$x^{\Delta}(t) := \lim\limits_{\mu^*(t) \searrow \mu(t)} \dfrac{x(\sigma(t)) - x(t)}{\mu^*(t)}$

We refer to the continuous and forced discrete logistic equations. The continuous logistic equation is of the form

$$\dot{x} = x(1-x).$$

The forced discrete logistic equation is given by

$$x_{k+1} = r_k x_k (1-x_k).$$

The standard discrete logistic equation is a special case of the forced discrete logistic equation where the sequence $\{r_k\}$ is a constant sequence. The chaotic behavior of the standard discrete logistic equation is well-studied [17], [18]. The chaotic behavior of the forced discrete logistic equation has been studied in the case where the sequence $\{r_k\}$ is periodic and consists of two values, A and B [19], [20]. Fractal images called Markus-Lyapunov fractals can be generated by producing a heat map of the Lyapunov exponents of the system in the A-B plane.

II. MAIN EXAMPLE

We consider the dynamic equation

$$x^{\Delta} = x(1-x) \tag{1}$$

on the pulse time scale $\mathbb{P}_{a,b_1,b_2;n}$. If we fix $a > 0$, in the limit as b_1 and b_2 approach 0, the time scale approaches the real numbers, and (1) becomes the standard continuous logistic equation. If we fix $b_1 = b_2 = \mu$, in the limit as a approaches 0, the time scale approaches $\mu\mathbb{Z}$, and (1) becomes

$$x^{\Delta} = \frac{x_{k+1} - x_k}{\mu} = x_k(1-x_k),$$

or,

$$x_{k+1} = x_k(\mu + 1 - \mu x_k). \tag{2}$$

While this does not seem to be of the form of the standard discrete logistic equation, we now show that (2) and the standard logistic equation are topologically conjugate.

Lemma 1. *The family of discrete time logistic maps is topologically conjugate to the family of logistic equations on a uniform time scale $\mu\mathbb{Z}$, where $\mu > 0$, as defined in* (2).

Proof: For $\mu > 0$, $f(x) = x(\mu+1-\mu x)$ and $g(x) = (\mu+1)x(1-x)$ are topologically conjugate via the homeomorphism $h(x) = (\mu/(\mu+1))x$. This follows since

$$\begin{aligned} h(f(x)) &= \frac{\mu}{\mu+1} x(\mu + 1 - \mu x) \\ &= \mu x \left(1 - \frac{\mu}{\mu+1} x\right) \\ &= (\mu+1)\frac{\mu}{\mu+1} x \left(1 - \frac{\mu}{\mu+1} x\right) \\ &= g(h(x)). \end{aligned}$$

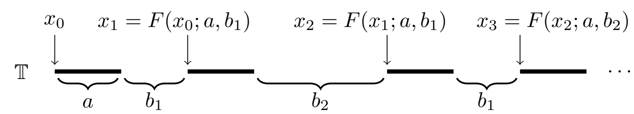

Fig. 1. Generating a sequence of representative values from an alternating pulse time scale, $\mathbb{P}_{a,b_1,b_2;1}$

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As the sign of the Lyapunov exponent for unimodal maps on a closed interval is preserved under topological conjugacy, [21] with Lemma 1, we see that the problem we are considering subsumes the cases of the discrete and continuous logistic equation, but is not limited to these cases.

The discrete-time logistic equation is chaotic for certain values of μ, but the continuous-time logistic equation is not chaotic. It is therefore natural to ask how the chaotic behavior emerges. Because both $\mathbb{R}$ and $\mu\mathbb{Z}$ are limiting cases of time scale of $\mathbb{P}_{a,b_1,b_2;n}$, we use the family of alternating pulse time scales as a natural way to examine the cases that are neither purely continuous nor purely discrete.

Chaos is a fairly new topic of study in the time scales literature [22], so we use a definition of chaos that is specific to $\mathbb{P}_{a,b_1,b_2;n}$. In the future we hope to find a definition of chaos that is consistent with arbitrary time scales.

Definition 3. *We say that the dynamic equation $x^{\Delta} = f(x)$ on the time scale $\mathbb{P}_{a,b_1,b_2;n}$ is chaotic if the sequence generated by evaluating the solution of the dynamic equation at the left endpoint of each successive continuous interval in the time scale is a chaotic sequence.*

In light of Definition 3, we seek a difference equation which generates the solution of (1) evaluated at the left endpoint of each continuous interval. We find the solution of (1) evaluated at the left endpoint of the next continuous interval after a gap of size b obeys the difference equation

$$x_{n+1} = \frac{e^a x_n\big((e^a-1)x_n - bx_n + b + 1\big)}{((e^a-1)x_n + 1)^2} := F(x_n; a, b). \tag{3}$$

One can arrive at the solution at the left endpoints of the closed intervals in $\mathbb{P}_{a,b_1,b_2;n}$ using the map F with different values of b. This is shown in Figure 1. Additionally, using standard tools in discrete chaos theory [18], we can easily calculate the Lyapunov exponent of the solution evaluated at the left endpoints.

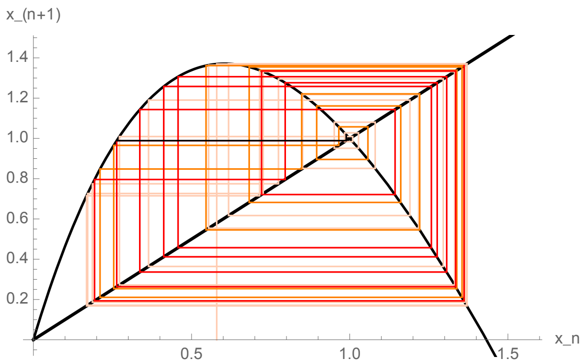

Fig. 2. Cobweb plot of the dynamics of (1) on a pulse time scale $\mathbb{P}_{.23,3.17}$ with an initial value $x_0 = 0.58$.

For example, consider the case where $b_1 = b_2 := b$, that is, when the time scale is $\mathbb{P}_{a,b}$. In this case, we see that certain values of a and b, such as $a = 1$, $b = 2$, the Lyapunov exponent of F is negative, and hence the system is not chaotic. Meanwhile, when $a = 1$, $b = 5$, the the Lyapunov exponent of F is positive and hence the system is chaotic.

III. Fractal Structure

We now show that fractal structure in the Lyapunov exponents exists within the b_1-b_2 space. To generate this fractal, we require bounds on the values of b for a fixed a where the Lyapunov exponent of the system can possibly be positive.

For the analysis of the upper bound, we examine the map F defined in Equation (3) in the x_n-x_{n+1} plane, as motivated by the cobweb plot in Figure 2. For the solution to stay in the first quadrant of the plane, we require that the maximum output of F is less than or equal to the x-intercept of F. The maximum output of F can be found to be equal to

$$M(b; a) = \frac{b+1}{e^a b - e^a + b + 1},$$

while the positive x-intercept of F is given by

$$c(b; a) = \frac{b+1}{-e^a + b + 1}.$$

Setting $M(b; a) = c(b; a)$, we see that for a fixed a, there are three solutions. Of these three solutions, one is positive. The positive solution of the equation $M(b; a) = c(b; a)$ gives us an upper bound

$$U(a) = \frac{\sqrt{e^{2a} + 8e^a} + e^a + 2}{2}.$$

For the analysis of the lower bound, we are motivated by the bifurcation diagram for the system. It is well-known that a period-three solution implies chaos. We can see in the bifurcation diagram of Figure 3 that period three cycles do exist. Finding the minimal value of b for which a period-three solution emerges is challenging [17]. Instead, we solve for the location of the first bifurcation as a sufficient lower bound. To do so, we solve $F^2(x; a, b) = x$. The two positive, non-constant solutions are conjugate with the radicand

$$\begin{aligned} h(a, b) = {} & e^{2a}b^4 + 2e^a b^3 - 2e^{3a}b^3 - 2e^a b^2 - 2e^{2a}b^2 - 2e^{3a}b^2 \\ & + e^{4a}b^2 + 2e^{4a}b - 2e^{2a} + e^{4a} + b^2 - 2b + 1. \end{aligned}$$

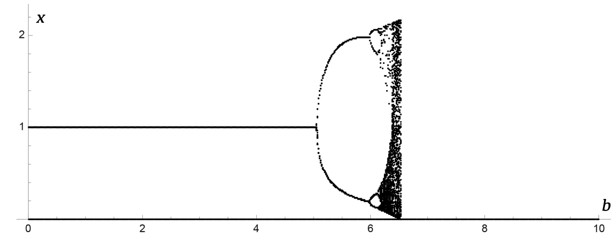

Fig. 3. Bifurcation diagram for $a = 1.4$. Notice that the first bifurcation occurs at $b = e^a + 1 \approx 5.055$.

The first bifurcation occurs when the radicand changes sign from negative to positive. The location of the first bifurcation, and hence the lower bound, is given by

$$L(a) = e^a + 1.$$

Given a specific value of a, there is a class of pulse time scales $\mathbb{P}_{a,b}$, with $L(a) \leq b \leq U(a)$, where the Lyapunov exponent of (1) can possibly be positive. To generate fractal images, we extend these bounds to both b_1 and b_2 for the alternating pulse time scale $\mathbb{P}_{a,b_1,b_2;n}$. Figure 4 shows a heat map for the Lyapunov exponent of (1) for two different values of a in the b_1-b_2 plane. It is worth mentioning that each point in the plane, and hence every pixel in the images in Figure 4, represents an alternating pulse time scale.

The Lyapunov exponent is computed numerically for each pixel in the images in Figure 4. Given $\mathbb{P}_{a,b_1,b_2;n}$, the value of x_N is computed, for $N = 120$. We then use the next $p = 12000$ values of $\{x_k\}$ to compute the Lyapunov exponent. These values of N and p produce high-fidelity images.

The colors in these images correspond to a heatmap based on the Lyapunov exponent of the time scale in that pixel. Positive Lyapunov exponents, and hence chaotic systems, are shaded in blue. Negative Lyapunov exponents close to zero are bright yellow. The heatmap darkens to dark red as Lyapunov exponents decrease. Regions of the same color in the non-chaotic region represent time scales with the same Lyapunov exponent and form iso-curves in the image.

IV. Conclusion

The same dynamic equation can have entirely different behavior with a small change in the time domain $\mathbb{T}$. This has implications for systems which may experience fluctuations in timing. There is a fractal structure to the qualitative behavior of the solution depending on the underlying time domain. We are able to see this fractal structure in a heat map of the Lyapunov exponent of the logistic equation on the time scale $\mathbb{P}_{a,b_1,b_2;n}$ in the $b_1 - b_2$ plane.

The fractal images produced in this paper are similar to the Markus-Lyapunov fractal. Indeed, the family of Markus-Lyapunov fractals are equivalent, in the sense of topological conjugacy, to the family of fractals described in this paper in the limit as $a \to 0^+$. In this light, our work expands upon the work of [20] with a few notable improvements. We make an array of one million values rather than 5000, with equidistant arguments on the interval from $L(a)$ to $U(a)$. Whereas some

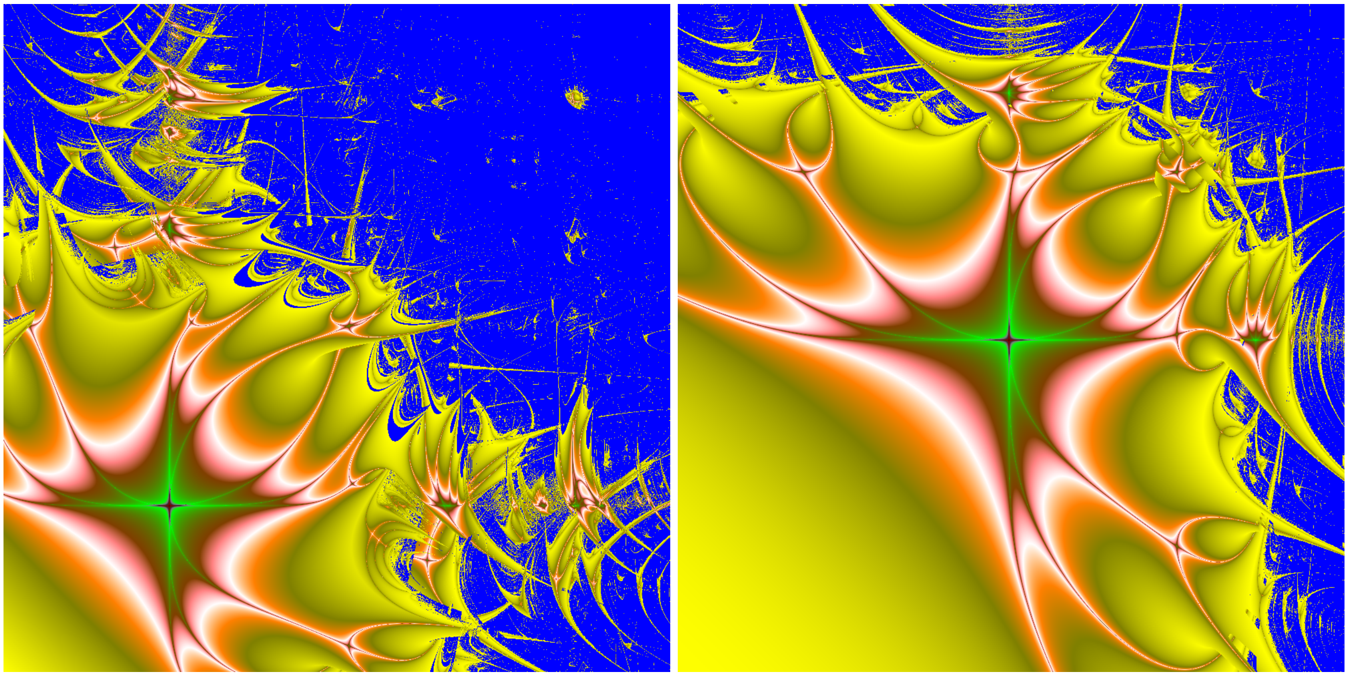

Fig. 4. The heat map of the Lyapunov exponent for system (1) on the time scale $\mathbb{P}_{a,b_1,b_2;6}$ in the b_1-b_2 plane with $L(a) \leq b_1, b_2 \leq U(a)$. In the left image, $a = 0.1$, while in the right image, $a = 4.9$.

pixels in the previous work are calculated by interpolation, each pixel in our image is computed directly. We use 12000 iterations for N in the Lyapunov exponent calculation, although fewer iterations may still produce visually pleasing results. Finally, we compute 120 beginning iterations of x_n before computing the exponent to allow fleeting x_n values to be discarded. Since we compute more iterations, we more-accurately represent the chaotic region than the previous work.

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